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An Algebra Refresher

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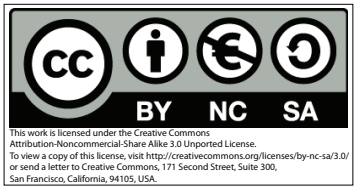

Contents

Foreword

The material in this refresher course has been designed to enable you to prepare for your university mathematics programme. When your programme starts you will find that your ability to get the best from lectures and tutorials, and to understand new material, depends crucially upon having a good facility with algebraic manipulation. We think that this is so important that we are making this course available for you to work through before you come to university.

How to use this booklet

First of all, hide your calculator, and don't use it at all during this exercise!

You are advised to work through each section in this booklet in order. You may need to revise some topics by looking at a GCSE or A-level textbook which contains information about algebraic techniques.

You should attempt a range of questions from each section, and check your answers with those at the back of the booklet. We have left sufficient space in the booklet so that you can do any necessary working within it. So, treat this as a work-book.

If you get questions wrong, or find topics unfamiliar, you can access quick reference help leaflets from the mathcentre site by following the links in the document.

If you cannot sort out your difficulties, do not worry about this. There will be provision to help you when you start your university course. This may take the form of special revision lectures, self-study revision material or a drop-in mathematics support centre.

Level

This material has been prepared for students who have completed an A-level course in mathematics.

Reminders

Use this page to note topics and questions which you found difficult.

Remember - seek help with these as soon as you start your university course.

1. Arithmetic of fractions

1. Express each of the following as a fraction in its simplest form. For example $\frac{3}{21}$ can be written as $\frac{1}{7}$. Remember, no calculators!

a) $\frac{20}{45}$ b) $\frac{16}{36}$ c) $-\frac{42}{21}$ d) $\frac{18}{16}$ e) $\frac{30}{30}$ f) $\frac{17}{21}$ g) $-\frac{49}{35}$ h) $\frac{90}{30}$

2. Add or subtract the following fractions as indicated.

a) $\frac{1}{2}+\frac{1}{3}$ b) $\frac{1}{2}-\frac{1}{3}$ c) $\frac{2}{3}+\frac{3}{4}$ d) $\frac{5}{6}-\frac{2}{3}$ e) $\frac{8}{9}+\frac{1}{5}+\frac{1}{6}$ f) $\frac{4}{5}+\frac{3}{7}-\frac{9}{10}$

3. Multiply the following fractions expressing each answer in its simplest form.

a) $\frac{4}{5}\times\frac{3}{16}$ b) $2\times3\times\frac{1}{4}$ c) $\frac{3}{4}\times\frac{3}{4}$ d) $\frac{4}{9}\times6$ e) $\frac{15}{16}\times\frac{4}{5}$ f) $\frac{9}{5}\times\frac{1}{3}\times\frac{15}{27}$

4. Divide the following fractions as indicated.

a) $3 \div \frac{1}{2}$ b) $\frac{1}{2} \div \frac{1}{4}$ c) $\frac{6}{7} \div \frac{16}{21}$ d) $\frac{\frac{3}{4}}{4}$ e) $5 \div \frac{10}{9}$ f) $\frac{3}{4} \div \frac{4}{3}$

5. Express the following as mixed fractions A mixed fraction has a whole number part and a fractional part. For example, $\frac{13}{5}$ can be written as the mixed fraction $2\frac{3}{5}$.

a) $\frac{5}{2}$ b) $\frac{7}{3}$ c) $-\frac{11}{4}$ d) $\frac{6}{5}$ e) $\frac{12}{5}$ f) $\frac{18}{7}$ g) $\frac{16}{3}$ h) $\frac{83}{9}$

6. Express the following as improper fractions. An improper fraction is 'top-heavy'. Its numerator is greater than its denominator. For example, the mixed fraction $13\frac{4}{5}$ can be written as the improper fraction $\frac{69}{5}$.

a) $2\frac{1}{4}$ b) $3\frac{1}{2}$ c) $5\frac{2}{3}$ d) $-3\frac{2}{5}$ e) $11\frac{4}{6}$ f) $8\frac{2}{9}$ g) $16\frac{3}{4}$ h) $89\frac{2}{7}$

2. Manipulation of expressions involving indices

1. Simplify the following algebraic expressions using the laws of indices.

a) $x^3 \times x^4$ b) $y^2 \times y^3 \times y^5$ c) $z^3 \times z^2 \times z$ d) $t^2 \times t^{10} \times t$

e) $a \times a \times a^2$ f) $t^3 t^4$ g) $b^6 b^3 b$ h) $z^7 z^7$

2. Simplify the following algebraic expressions using the laws of indices.

a) $\dfrac{x^6}{x^2}$ b) $\dfrac{y^{14}}{y^{10}}$ c) $\dfrac{t^{16}}{t^{12}}$ d) $\dfrac{z^{10}}{z^9}$ e) $\dfrac{v^7}{v^0}$ f) x^7/x^4

3. Simplify the following algebraic expressions using the laws of indices.

a) $\dfrac{10^7}{10^6}$ b) $\dfrac{10^{19}}{10^{16}}$ c) $\dfrac{x^7}{x^{14}}$ d) $\dfrac{x^7}{y^4}$

e) $\dfrac{(ab)^4}{a^2b^2}$ f) $\dfrac{9^9 10^{10}}{10^9}$ g) $\dfrac{x^9y^8}{y^7x^6}$, h) $\dfrac{(abc)^3}{(abc)^2}$

4. Write the following expressions using only *positive* indices. For example $\dfrac{x^{-4}}{x^{-2}}$ can be written as $\dfrac{1}{x^2}$.

a) $x^{-2}x^{-1}$ b) $\dfrac{3x}{x^{-4}}$ c) $\dfrac{t^{-2}}{t^{-3}}$

d) $(2a^2b^3)(6a^{-3}b^{-5})$ e) $\dfrac{x^{-3}}{5^{-2}}$ f) $\dfrac{(27)^{-1}x^{-1}}{y^{-2}}$

5. Without using a calculator, evaluate

a) $\dfrac{3}{4^{-2}}$ b) 4×3^{-2} c) $3^{-1}9^2(27)^{-1}$

d) $(0.25)^{-1}$ e) $(0.2)^{-2}$ f) $(0.1)^{-3}$

6. Simplify

a) $t^{-6}t^3$ b) $(-y^{-2})(-y^{-1})$ c) $\dfrac{3y^{-2}}{6y^{-3}}$ d) $(-2t^{-1})(-3t^{-2})(-4t^{-3})$

e) $\dfrac{3t^{-2}}{6t^3}$ f) $\dfrac{(2t^{-1})^3}{6t^2}$ g) $\dfrac{(-2t)^3}{(-4t)^2}$

7. Write the following expressions using a *single* index. For example $(5^3)^{-4}$ can be written as 5^{-12}.

a) $(5^3)^5$ b) $(3^3)^3$ c) $(17^2)^4$ d) $(y^3)^6$ e) $(\frac{y^{-1}}{y^{-2}})^3$

f) $(\frac{t^{-2}}{t^4})^3$ g) $(k^{-2})^{-6}$ h) $((-1)^4)^3$ i) $((-1)^{-4})^{-3}$

8. Without the use of a calculator, evaluate

a) $(4^{-1})^2$ b) $(2^2)^{-1}$ c) $(3^2)^2$ d) $(6^{-2})^{-1}$

e) $\left(\frac{2}{5^2}\right)^{-1}$ f) $(-2)^{-1}$ g) $\left(-\frac{2}{3}\right)^{-2}$

9. Write the following expressions without using brackets.

a) $(4^2 5^3)^3$ b) $\left(\frac{3ab}{c^3}\right)^2$ c) $\left(\frac{4^{-2}a^{-3}}{b^{-1}}\right)^2$ d) $(2a^2b)^3$

e) $(3xy^2z^3)^2$ f) $\left(\frac{6}{ab^2}\right)^2$ g) $\left(-\frac{3}{x^2}\right)^2$ h) $\left(\frac{2z^2}{3t}\right)^3$

i) $(-2x)^2$ j) $(-2x^2)^{-2}$ k) $\left(-\frac{2}{x^2}\right)^{-3}$

10. Write the following expressions without using brackets.

a) $(6^{1/2})^3$ b) $(5^{1/3})^6$ c) $(10^{0.6})^4$ d) $(x^2)^{1/3}$

e) $(2x^2)^{1/3}$ f) $(a \times a^2)^{1/2}$ g) $(ab^2)^{1/2}$

11. Write the following expressions without using brackets.

a) $(4^3)^{-1/2}$ b) $(3^{-1/2})^{-1/2}$ c) $(7^{2/3})^4$ d) $(19^{3/2})^{1/3}$

e) $(a^2b^{-3})^{-\frac{3}{2}}$ f) $\left(\dfrac{k^{-1.5}}{\sqrt{k}}\right)^{-2}$

12. Write the following expressions without using brackets.

a) $(5b)^{1/6}$ b) $(3\sqrt{x})^3$ c) $3(\sqrt{x})^3$ d) $(\sqrt{3x})^3$

13. Simplify

a) $x^{1/2}x^{1/3}$ b) $\dfrac{x^{1/2}}{x^{1/3}}$ c) $(x^{1/2})^{1/3}$ d) $(8x^3)^{1/3}$

e) $\sqrt{25y^2}$ f) $\left(\frac{27}{t^3}\right)^{1/3}$ g) $(16y^4)^{1/4}$ h) $\left(x^{1/4}\,x^{1/2}\right)^4$

i) $\sqrt{a^2a^6}$ j) $\sqrt{\dfrac{a^{-4}}{a^{-1}}}$

3. Removing brackets and factorisation.

1. Write the following expressions without using brackets:

a) $2(mn)$ b) $2(m+n)$ c) $a(mn)$ d) $a(m+n)$ e) $a(m-n)$

f) $(am)n$ g) $(a+m)n$ h) $(a-m)n$ i) $5(pq)$ j) $5(p+q)$

k) $5(p-q)$ l) $7(xy)$ m) $7(x+y)$ n) $7(x-y)$ o) $8(2p+q)$

p) $8(2pq)$ q) $8(2p-q)$ r) $5(p-3q)$ s) $5(p+3q)$ t) $5(3pq)$

2. Write the following expressions without using brackets and simplify where possible:

a) $(2+a)(3+b)$ b) $(x+1)(x+2)$ c) $(x+3)(x+3)$ d) $(x+5)(x-3)$

3. Write the following expressions without using brackets:

a) $(7+x)(2+x)$ b) $(9+x)(2+x)$ c) $(x+9)(x-2)$ d) $(x+11)(x-7)$

e) $(x+2)x$ f) $(3x+1)x$ g) $(3x+1)(x+1)$ h) $(3x+1)(2x+1)$

i) $(3x+5)(2x+7)$ j) $(3x+5)(2x-1)$ k) $(5-3x)(x+1)$ l) $(2-x)(1-x)$

4. Rewrite the following expressions without using brackets:

a) $(s+1)(s+5)(s-3)$ b) $(x+y)^3$

5. Factorise

a) $5x+15y$ b) $3x-9y$ c) $2x+12y$ d) $4x+32z+16y$ e) $\frac{1}{2}x+\frac{1}{4}y$

6. Factorise

a) $\frac{1}{3}x + \frac{1}{6}xy$ b) $\frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$ c) $a^2 - a + \frac{1}{4}$ d) $\frac{1}{x^2} - \frac{2}{x} + 1$

7. Factorise

a) $x^2 + 8x + 7$ b) $x^2 + 6x - 7$ c) $x^2 + 7x + 10$ d) $x^2 - 6x + 9$ e) $x^2 + 5x + 6$.

8. Factorise

a) $2x^2 + 3x + 1$ b) $2x^2 + 4x + 2$ c) $3x^2 - 3x - 6$ d) $5x^2 - 4x - 1$
e) $16x^2 - 1$ f) $-x^2 + 1$ g) $-2x^2 + x + 3$

9. Factorise

a) $x^2 + 9x + 14$ b) $x^2 + 11x + 18$ c) $x^2 + 7x - 18$ d) $x^2 + 4x - 77$

e) $x^2 + 2x$ f) $3x^2 + x$, g) $3x^2 + 4x + 1$ h) $6x^2 + 5x + 1$

i) $6x^2 + 31x + 35$ j) $6x^2 + 7x - 5$ k) $-3x^2 + 2x + 5$ l) $x^2 - 3x + 2$

10. Rewrite the following expressions without using brackets, simplifying where possible:

a) $15 - (7 - x)$ b) $15 - 7(1 - x)$

c) $15 - 7(x - 1)$ d) $(2x - y) - x(1 + y)$

e) $x(a + b) - x(a + 3b)$ f) $2(5a + 3b) + 3(a - 2b)$

g) $-(4a + 5b - 3c) - 2(2a + 3b - 4c)$ h) $2x(x - 5) - x(x - 2) - 3x(x - 5)$

11. Rewrite each of the following expressions without using brackets and simplify where possible

a) $2x - (3y + 8x)$, b) $2x + 5(x - y - z)$, c) $-(5x - 3y)$, d) $5(2x - y) - 3(x + 2y)$

4. Arithmetic of algebraic fractions

1. Express each of the following as a single fraction.

a) $2 \times \dfrac{x+y}{3}$ b) $\dfrac{1}{3} \times 2(x+y)$ c) $\dfrac{2}{3} \times (x+y)$

2. Simplify

a) $3 \times \dfrac{x+4}{7}$ b) $\dfrac{1}{7} \times 3(x+4)$ c) $\dfrac{3}{7} \times (x+4)$ d) $\dfrac{x}{y} \times \dfrac{x+1}{y+1}$

e) $\dfrac{1}{y} \times \dfrac{x^2+x}{y+1}$ f) $\dfrac{\pi d^2}{4} \times \dfrac{Q}{\pi d^2}$ g) $\dfrac{Q}{\pi d^2/4}$ h) $\dfrac{1}{x/y}$

3. Simplify a) $\dfrac{6/7}{s+3}$ b) $\dfrac{3/4}{x-1}$ c) $\dfrac{x-1}{3/4}$

4. Simplify $\dfrac{3}{x+2} \div \dfrac{x}{2x+4}$

5. Simplify $\dfrac{5}{2x+1} \div \dfrac{x}{3x-1}$

6. Simplify

a) $\dfrac{x}{4} + \dfrac{x}{7}$ b) $\dfrac{2x}{5} + \dfrac{x}{9}$ c) $\dfrac{2x}{3} - \dfrac{3x}{4}$ d) $\dfrac{x}{x+1} - \dfrac{2}{x+2}$

e) $\dfrac{x+1}{x} + \dfrac{3}{x+2}$ f) $\dfrac{2x+1}{3} - \dfrac{x}{2}$ g) $\dfrac{x+3}{2x+1} - \dfrac{x}{3}$ h) $\dfrac{x}{4} - \dfrac{x}{5}$

7. Simplify

a) $\dfrac{1}{x+2} + \dfrac{2}{x+3}$ b) $\dfrac{2}{x+3} + \dfrac{5}{x+1}$ c) $\dfrac{2}{2x+1} - \dfrac{3}{3x+2}$

d) $\dfrac{x+1}{x+3} + \dfrac{x+4}{x+2}$ e) $\dfrac{x-1}{x-3} + \dfrac{x-1}{(x-3)^2}$

8. Express as a single fraction $\dfrac{1}{7}s + \dfrac{11}{21}$

9. Express $\dfrac{A}{2x+3}+\dfrac{B}{x+1}$ as a single fraction.

10. Express $\dfrac{A}{2x+5}+\dfrac{B}{(x-1)}+\dfrac{C}{(x-1)^2}$ as a single fraction.

11. Express $\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}$ as a single fraction.

12. Express $\dfrac{Ax+B}{x^2+x+10}+\dfrac{C}{x-1}$ as a single fraction.

13. Express $Ax+B+\dfrac{C}{x+1}$ as a single fraction.

14. Show that $\dfrac{x_1}{\frac{1}{x_3}-\frac{1}{x_2}}$ is equal to $\dfrac{x_1x_2x_3}{x_2-x_3}$.

15. Simplify a) $\dfrac{3x}{4}-\dfrac{x}{5}+\dfrac{x}{3}$, b) $\dfrac{3x}{4}-\left(\dfrac{x}{5}+\dfrac{x}{3}\right)$.

16. Simplify $\dfrac{5x}{25x+10y}$.

17. Simplify $\dfrac{x+2}{x^2+3x+2}$.

18. Explain why no cancellation is possible in the expression $\dfrac{a+2b}{a-2b}$.

19. Simplify $\dfrac{x+2}{x^2+9x+20} \times \dfrac{x+5}{x+2}$

20. Simplify $\dfrac{5}{7y} + \dfrac{2x}{3}$.

21. Express as single fraction $\dfrac{3}{x-4} - \dfrac{2}{(x-4)^2}$.

22. Express as a single fraction $2x - 1 + \dfrac{4}{x} + \dfrac{3}{2x+1}$.

23. a) Express $\dfrac{1}{u} + \dfrac{1}{v}$ as a single fraction. b) Hence find the reciprocal of $\dfrac{1}{u} + \dfrac{1}{v}$.

24. Express $\dfrac{1}{s}+\dfrac{1}{s^2}$ as a single fraction.

25. Express $-\dfrac{6}{s+3}-\dfrac{4}{s+2}+\dfrac{3}{s+1}+2$ as a single fraction.

26. State which of the following expressions are equivalent to

$$\frac{2x+1}{2x+4}+\frac{x}{2}$$

a) $\dfrac{x+1}{x+4}+\dfrac{x}{2}$ b) $\dfrac{3x+1}{2x+6}$ c) $1+\dfrac{x}{2}-\dfrac{3}{2x+4}$

d) $\dfrac{2x+1}{2x}+\dfrac{2x+1}{4}+\dfrac{x}{2}$ e) $\dfrac{x^2+4x+1}{2(x+2)}$ f) $1+\dfrac{1}{4}+\dfrac{x}{2}$

5. Surds

Roots, for example $\sqrt{2}$, $\sqrt{5}$, $\sqrt[3]{6}$ are also known as **surds**. A common cause of error is misuse of expressions involving surds. You should be aware that $\sqrt{ab} = \sqrt{a}\sqrt{b}$ but $\sqrt{a+b}$ is NOT equal to $\sqrt{a} + \sqrt{b}$.

1. It is often possible to write surds in equivalent forms. For example $\sqrt{48}$ can be written $\sqrt{3 \times 16} = \sqrt{3} \times \sqrt{16} = 4\sqrt{3}$.

Write the following in their simplest forms:

a) $\sqrt{180}$ b) $\sqrt{63}$

2. By multiplying numerator and denominator by $\sqrt{2}+1$ show that

$$\frac{1}{\sqrt{2}-1} \quad \text{is equivalent to} \quad \sqrt{2}+1$$

3. Simplify, if possible, a) $\sqrt{x^2y^2}$ b) $\sqrt{x^2+y^2}$.

4. Study the following expressions and simplify where possible.

a) $\sqrt{(x+y)^4}$ b) $(\sqrt[3]{x+y})^6$ c) $\sqrt{x^4+y^4}$

5. By considering the expression $(\sqrt{x}+\sqrt{y})^2$ show that

$$\sqrt{x}+\sqrt{y}=\sqrt{x+y+2\sqrt{xy}}$$

Find a corresponding expression for $\sqrt{x}-\sqrt{y}$.

6. Write each of the following as an expression under a single square root sign. (For parts c) and d) see Question 5 above.)

a) $2\sqrt{p}$ b) $\sqrt{p}\sqrt{q^3}$ c) $\sqrt{p}+\sqrt{2q}$ d) $\sqrt{3}-\sqrt{2}$

7. Use indices (powers) to write the following expressions without the root sign.

a) $\sqrt[4]{a^2}$ b) $(\sqrt{3}\times\sqrt{5})^3$

6. Solving linear equations.

In questions $1-35$ solve each equation:

1. $3y-8=\frac{1}{2}y$ **2.** $7t-5=4t+7$ **3.** $3x+4=4x+3$ **4.** $4-3x=4x+3$

5. $3x+7=7x+2$ **6.** $3(x+7)=7(x+2)$ **7.** $2x-1=x-3$ **8.** $2(x+4)=8$

9. $-2(x-3)=6$ **10.** $-2(x-3)=-6$

11. $-3(3x-1)=2$ **12.** $2-(2t+1)=4(t+2)$

13. $5(m-3)=8$ **14.** $5m-3=5(m-3)+2m$

15. $2(y+1)=-8$ **16.** $17(x-2)+3(x-1)=x$

17. $\frac{1}{3}(x+3) = -9$ **18.** $\frac{3}{m} = 4$ **19.** $\frac{5}{m} = \frac{2}{m+1}$ **20.** $-3x + 3 = 18$

21. $3x + 10 = 31$ **22.** $x + 4 = \sqrt{8}$ **23.** $x - 4 = \sqrt{23}$

24. $\frac{x-5}{2} - \frac{2x-1}{3} = 6$ **25.** $\frac{x}{4} + \frac{3x}{2} - \frac{x}{6} = 1$

26. $\frac{x}{2} + \frac{4x}{3} = 2x - 7$ **27.** $\frac{5}{3m+2} = \frac{2}{m+1}$

28. $\dfrac{2}{3x-2} = \dfrac{5}{x-1}$ **29.** $\dfrac{x-3}{x+1} = 4$ **30.** $\dfrac{x+1}{x-3} = 4$ **31.** $\dfrac{y-3}{y+3} = \dfrac{2}{3}$

32. $\dfrac{4x+5}{6} - \dfrac{2x-1}{3} = x$ **33.** $\dfrac{3}{2s-1} + \dfrac{1}{s+1} = 0$

34. $\dfrac{1}{5x} + \dfrac{1}{4x} = 10.$ **35.** $\dfrac{3}{s-1} = \dfrac{2}{s-5}.$

7. Transposition of formulae

1. Make t the subject of the formula $p = \dfrac{c}{\sqrt{t}}$.

2. Make N the subject of the formula $L = \dfrac{\mu N^2 A}{\ell}$.

3. In each case make the specified variable the subjectof the formula:

a) $h = c + d + 2e$, e b) $S = 2\pi r^2 + 2\pi rh$, h

c) $Q = \sqrt{\dfrac{c+d}{c-d}}$, c d) $\dfrac{x+y}{3} = \dfrac{x-y}{7} + 2$, x

4. Make n the subjectof the formula $J = \dfrac{nE}{nL + m}$.

8. Solving quadratic equations by factorisation

Solve the following equations by factorisation:

1. $x^2-3x+2=0$ **2.** $x^2-x-2=0$ **3.** $x^2+x-2=0$ **4.** $x^2+3x+2=0$

5. $x^2+8x+7=0$ **6.** $x^2-7x+12=0$ **7.** $x^2-x-20=0$ **8.** $x^2-1=0$

9. $x^2-2x+1=0$ **10.** $x^2+2x+1=0$ **11.** $x^2+11x=0$ **12.** $2x^2+2x=0$

13. $x^2-3x=0$ **14.** $x^2+9x=0$ **15.** $2x^2-5x+2=0$ **16.** $6x^2-x-1=0$

17. $-5x^2+6x-1=0$ **18.** $-x^2+4x-3=0$

9. Solving quadratic equations:

using a standard formula and by completing the square

Solve each of the following quadratic equations twice: once by using the formula, then again by completing the square. Obtain your answers in surd, not decimal, form.

1. $x^2 + 8x + 1 = 0$ **2.** $x^2 + 7x - 2 = 0$ **3.** $x^2 + 6x - 2 = 0$

4. $4x^2 + 3x - 2 = 0$ **5.** $2x^2 + 3x - 1 = 0$ **6.** $x^2 + x - 1 = 0$

7. $-x^2 + 3x + 1 = 0$ **8.** $-2x^2 - 3x + 1 = 0$ **9.** $2x^2 + 5x - 3 = 0$

10. $-2s^2 - s + 3 = 0$ **11.** $9x^2 + 16x + 1 = 0$ **12.** $x^2 + 16x + 9 = 0$

13. Show that the roots of $x^2 - 2x + \alpha = 0$ are $x = 1 + \sqrt{1 - \alpha}$ and $x = 1 - \sqrt{1 - \alpha}$.

14. Show that the roots of $x^2 - 2\alpha x + \beta = 0$ are

$$x = \alpha + \sqrt{\alpha^2 - \beta} \quad \text{and} \quad x = \alpha - \sqrt{\alpha^2 - \beta}$$

15. Show that the roots of $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$ are

$$\lambda = \frac{(a + d) \pm \sqrt{a^2 + d^2 - 2ad + 4bc}}{2}$$

10. Solving some polynomial equations

1. Factorise $x^3 - x^2 - 65x - 63$ given that $(x + 7)$ is a factor.

2. Show that $x = -1$ is a root of $x^3 + 11x^2 + 31x + 21 = 0$ and locate the other roots algebraically.

3. Show that $x = 2$ is a root of $x^3 - 3x - 2 = 0$ and locate the other roots.

4. Solve the equation $x^4 - 2x^2 + 1 = 0$.

5. Factorise $x^4 - 7x^3 + 3x^2 + 31x + 20$ given that $(x+1)^2$ is a factor.

6. Given that two of the roots of $x^4 + 3x^3 - 7x^2 - 27x - 18 = 0$ have the same modulus but different sign, solve the equation.

(Hint - let two of the roots be α and $-\alpha$ and use the technique of equating coefficients).

11. Partial Fractions

1.

a) Find the partial fractions of $\dfrac{5x-1}{(x+1)(x-2)}$.

b) Check your answer by adding the partial fractions together again.

c) Express in partial fractions: $\dfrac{7x+25}{(x+4)(x+3)}$.

d) Check your answer by adding the partial fractions together again.

2. Find the partial fractions of $\dfrac{11x+1}{(x-1)(2x+1)}$.

3. Express each of the following as the sum of partial fractions:

a) $\dfrac{3}{(x+1)(x+2)}$ b) $\dfrac{5}{x^2+7x+12}$ c) $\dfrac{-3}{(2x+1)(x-3)}$

4. Express the following in partial fractions.

a) $\dfrac{3-x}{x^2-2x+1}$ b) $-\dfrac{7x-15}{(x-1)^2}$ c) $\dfrac{3x+14}{x^2+8x+16}$

d) $\dfrac{5x+18}{(x+4)^2}$ e) $\dfrac{2x^2-x+1}{(x+1)(x-1)^2}$ f) $\dfrac{5x^2+23x+24}{(2x+3)(x+2)^2}$

g) $\dfrac{6x^2-30x+25}{(3x-2)^2(x+7)}$ h) $\dfrac{s+2}{(s+1)^2}$ i) $\dfrac{2s+3}{s^2}$.

5. Express each of the following as the sum of its partial fractions.

a) $\dfrac{3}{(x^2+x+1)(x-2)}$ b) $\dfrac{27x^2-4x+5}{(6x^2+x+2)(x-3)}$

c) $\dfrac{2x+4}{4x^2+12x+9}$ d) $\dfrac{6x^2+13x+2}{(x^2+5x+1)(x-1)}$

6. Express each of the following as the sum of its partial fractions.

a) $\dfrac{x+3}{x+2}$ b) $\dfrac{3x-7}{x-3}$ c) $\dfrac{x^2+2x+2}{x+1}$ d) $\dfrac{2x^2+7x+7}{x+2}$

e) $\dfrac{3x^5+4x^4-21x^3-40x^2-24x-29}{(x+2)^2(x-3)}$ f) $\dfrac{4x^5+8x^4+23x^3+27x^2+25x+9}{(x^2+x+1)(2x+1)}$

7. Express in partial fractions:

a) $\dfrac{2x-4}{x(x-1)(x-3)}$ b) $\dfrac{1+x}{(x+3)^2(x+1)}$ c) $\dfrac{x^2+1}{(2x+1)(x-1)(x-3)}$

d) $\dfrac{4s-3}{2s+1}$ e) $\dfrac{3s+1}{s(s-2)}$

8. Express in partial fractions

$$\frac{K(1+\alpha s)}{(1+\tau s)s}$$

where K, α and τ are constants.

9. Express in partial fractions partial fractions

a) $\dfrac{2s+1}{s^5(s+1)}$ b) $\dfrac{2s^3+6s^2+6s+3}{s+1}$

10. Express in partial fractions $\dfrac{2x+1}{(x-2)(x+1)(x-3)}$

Answers

Section 1. Arithmetic of fractions

1. a) $\frac{4}{9}$, b) $\frac{4}{9}$, c) -2, d) $\frac{9}{8}$, e) 1, f) $\frac{17}{21}$, g) $-\frac{7}{5}$, h) 3
2. a) $\frac{5}{6}$, b) $\frac{1}{6}$, c) $\frac{17}{12}$, d) $\frac{1}{6}$, e) $\frac{113}{90}$, f) $\frac{23}{70}$
3. a) $\frac{3}{20}$, b) $\frac{3}{2}$, c) $\frac{9}{16}$, d) $\frac{8}{3}$, e) $\frac{3}{4}$, f) $\frac{1}{3}$.
4. a) 6, b) 2, c) $\frac{9}{8}$, d) $\frac{3}{16}$, e) $\frac{9}{2}$, f) $\frac{9}{16}$
5. a) $2\frac{1}{2}$, b) $2\frac{1}{3}$, c) $-2\frac{3}{4}$, d) $1\frac{1}{5}$, e) $2\frac{2}{5}$, f) $2\frac{4}{7}$, g) $5\frac{1}{3}$, h) $9\frac{2}{9}$
6. a) $\frac{9}{4}$, b) $\frac{7}{2}$, c) $\frac{17}{3}$, d) $-\frac{17}{5}$, e) $\frac{35}{3}$, f) $\frac{74}{9}$, g) $\frac{67}{4}$, h) $\frac{625}{7}$

Section 2. Manipulation of expressions involving indices

1. a) x^7, b) y^{10}, c) z^6, d) t^{13}, e) a^4, f) t^7, g) b^{10}, h) z^{14}.
2. a) x^4, b) y^4, c) t^4, d) z, e) v^7, f) x^3
3. a) 10, b) 10^3, c) x^{-7}, d) $\frac{x^7}{y^4}$, e) a^2b^2, f) $9^9 \cdot 10$, g) x^3y, h) abc
4. a) $\frac{1}{x^3}$, b) $3x^5$, c) t, d) $\frac{12}{ab^2}$, e) $\frac{5^2}{x^3}$, f) $\frac{y^2}{27x}$
5. a) 48, b) $\frac{4}{9}$, c) 1, d) 4, e) 25, f) 1000
6. a) t^{-3}, b) y^{-3}, c) $\frac{1}{2}y$, d) $-24t^{-6}$, e) $\frac{1}{2t^5}$, f) $\frac{4t^{-5}}{3}$, g) $-\frac{t}{2}$.
7. a) 5^{15}, b) 3^9, c) 17^8, d) y^{18}, e) y^3, f) t^{-18}, g) k^{12}, h) $(-1)^{12} = 1$, i) $(-1)^{12} = 1$.
8. a) $\frac{1}{16}$, b) $\frac{1}{4}$, c) 81, d) 36, e) $\frac{25}{2}$, f) $-\frac{1}{2}$, g) $\frac{9}{4}$
9. a) $4^6 5^9$, b) $\frac{9a^2b^2}{c^6}$, c) $\frac{4^{-4}a^{-6}}{b^{-2}} = \frac{b^2}{4^4a^6}$ d) $8a^6b^3$, e) $9x^2y^4z^6$, f) $\frac{36}{a^2b^4}$, g) $\frac{9}{x^4}$, h) $\frac{8z^6}{27t^3}$, (i) $4x^2$, j) $\frac{1}{4x^4}$, k) $-\frac{x^6}{8}$.
10. a) $6^{3/2}$, b) 5^2, c) $10^{2.4}$, d) $x^{2/3}$, e) $2^{1/3}x^{2/3}$, f) $a^{3/2}$, g) $a^{1/2}b$.
11. a) $4^{-3/2}$, b) $3^{1/4}$, c) $7^{8/3}$, d) $19^{1/2}$, e) $a^{-3}b^{9/2}$, f) k^4.
12. a) $5^{1/6}b^{1/6}$, b) $27x^{3/2}$, c) $3x^{3/2}$, d) $3^{3/2}x^{3/2}$
13. a) $x^{5/6}$, b) $x^{1/6}$, c) $x^{1/6}$, d) $2x$, e) $5y$, f) $\frac{3}{t}$, g) $2y$, h) x^3, i) a^4, j) $a^{-3/2}$

Section 3. Removing brackets and factorisation

1. a) $2mn$, b) $2m + 2n$, c) amn, d) $am + an$, e) $am - an$, f) amn, g) $an + mn$, h) $an - mn$, i) $5pq$, j) $5p + 5q$, k) $5p - 5q$, l) $7xy$, m) $7x + 7y$, n) $7x - 7y$, o) $16p + 8q$, p) $16pq$, q) $16p - 8q$, r) $5p - 15q$, s) $5p + 15q$, t) $15pq$
2. a) $6 + 3a + 2b + ab$, b) $x^2 + 3x + 2$, c) $x^2 + 6x + 9$, d) $x^2 + 2x - 15$
3. a) $14 + 9x + x^2$, b) $18 + 11x + x^2$, c) $x^2 + 7x - 18$, d) $x^2 + 4x - 77$, e) $x^2 + 2x$, f) $3x^2 + x$, g) $3x^2 + 4x + 1$
   h) $6x^2 + 5x + 1$, i) $6x^2 + 31x + 35$, j) $6x^2 + 7x - 5$
   k) $-3x^2 + 2x + 5$, l) $x^2 - 3x + 2$
4. a) $s^3 + 3s^2 - 13s - 15$, b) $x^3 + 3x^2y + 3xy^2 + y^3$

5. a) $5(x+3y)$, b) $3(x-3y)$, c) $2(x+6y)$, d) $4(x+8z+4y)$, e) $\frac{1}{2}(x+\frac{1}{2}y)$

6. a) $\frac{x}{3}\left(1+\frac{y}{2}\right)$, b) $\frac{\pi r^2}{3}(2r+h)$, c) $\left(a-\frac{1}{2}\right)^2$, d) $\left(\frac{1}{x}-1\right)^2$.

7. a) $(x+7)(x+1)$, b) $(x+7)(x-1)$, c) $(x+2)(x+5)$, d) $(x-3)(x-3)=(x-3)^2$, e) $(x+3)(x+2)$

8. a) $(2x+1)(x+1)$, b) $2(x+1)^2$, c) $3(x+1)(x-2)$, d) $(5x+1)(x-1)$, e) $(4x+1)(4x-1)$, f) $(x+1)(1-x)$, g) $(x+1)(3-2x)$

9. a) $(7+x)(2+x)$, b) $(9+x)(2+x)$, c) $(x+9)(x-2)$, d) $(x+11)(x-7)$, e) $(x+2)x$, f) $(3x+1)x$, g) $(3x+1)(x+1)$, h) $(3x+1)(2x+1)$ i) $(3x+5)(2x+7)$, j) $(3x+5)(2x-1)$, k) $(5-3x)(x+1)$ l) $(2-x)(1-x)$

10. a) $8+x$, b) $8+7x$, c) $22-7x$, d) $x-y-xy$, e) $-2bx$, f) $13a$, g) $-8a-11b+11c$, h) $7x-2x^2$.

11. a) $-3y-6x$, b) $7x-5y-5z$, c) $-5x+3y$, d) $7x-11y$.

Section 4. Arithmetic of Algebraic Fractions

1. a) $\frac{2(x+y)}{3}$, b) $\frac{2(x+y)}{3}$, c) $\frac{2(x+y)}{3}$

2. a) $\frac{3(x+4)}{7}$, b) $\frac{3(x+4)}{7}$, c) $\frac{3(x+4)}{7}$, d) $\frac{x(x+1)}{y(y+1)}$, e) $\frac{x(x+1)}{y(y+1)}$, f) $Q/4$, g) $\frac{4Q}{\pi d^2}$, h) $\frac{y}{x}$.

3. a) $\frac{6}{7(s+3)}$, b) $\frac{3}{4(x-1)}$, c) $\frac{4(x-1)}{3}$.

4. $\frac{6}{x}$. 5. $\frac{5(3x-1)}{x(2x+1)}$

6. a) $\frac{11x}{28}$, b) $\frac{23x}{45}$, c) $-\frac{x}{12}$, d) $\frac{x^2-2}{(x+1)(x+2)}$, e) $\frac{x^2+6x+2}{x(x+2)}$, f) $\frac{x+2}{6}$, g) $\frac{9+2x-2x^2}{3(2x+1)}$, h) $\frac{x}{20}$.

7. a) $\frac{3x+7}{(x+2)(x+3)}$, b) $\frac{7x+17}{(x+3)(x+1)}$, c) $\frac{1}{(2x+1)(3x+2)}$, d) $\frac{2x^2+10x+14}{(x+3)(x+2)}$, e) $\frac{x^2-3x+2}{(x-3)^2}$

8. $\frac{3s+11}{21}$.

9. $\frac{A(x+1)+B(2x+3)}{(2x+3)(x+1)}$.

10. $\frac{A(x-1)^2+B(x-1)(2x+5)+C(2x+5)}{(2x+5)(x-1)^2}$

11. $\frac{A(x+1)+B}{(x+1)^2}$

12. $\frac{(Ax+B)(x-1)+C(x^2+x+10)}{(x-1)(x^2+x+10)}$

13. $\frac{(Ax+B)(x+1)+C}{x+1}$

15. a) $\frac{53x}{60}$, b) $\frac{13x}{60}$.

16. $\frac{x}{5x+2y}$, 17. $\frac{1}{x+1}$, 19. $\frac{1}{x+4}$, 20. $\frac{15+14xy}{21y}$,

21. $\frac{3x-14}{(x-4)^2}$, 22. $\frac{4x^3+10x+4}{2x^2+x}$. 23. a) $\frac{v+u}{uv}$, b) $\frac{uv}{v+u}$.

24. $\frac{s+1}{s^2}$. 25. $\frac{2s^3+5s^2+3s+6}{s^3+6s^2+11s+6}$. 26. c) and e) only. Note in particular that whilst $\frac{a+b}{c}$ is equal to $\frac{a}{c}+\frac{b}{c}$, $\frac{a}{b+c}$ is NOT equal to $\frac{a}{b}+\frac{a}{c}$, in general.

Section 5. Surds

1. a) $\sqrt{180} = \sqrt{36 \times 5} = 6\sqrt{5}$. b) $\sqrt{63} = \sqrt{9 \times 7} = 3\sqrt{7}$.
3. a) xy. b) Note that $\sqrt{x^2 + y^2}$ is NOT equal to $x + y$.
4. a) $(x + y)^2$, b) $(x + y)^2$, c) $\sqrt{x^4 + y^4}$ is NOT equal to $x^2 + y^2$.
5. $\sqrt{x} - \sqrt{y} = \sqrt{x + y - 2\sqrt{xy}}$.
6. a) $\sqrt{4p}$, b) $\sqrt{pq^3}$, c) $\sqrt{p + 2q + 2\sqrt{2pq}}$, d) $\sqrt{5 - 2\sqrt{6}}$.
7. a) $a^{1/2}$, b) $15^{3/2}$

Section 6. Solving linear equations

1. $16/5$,	2. 4,	3. 1,	4. $1/7$,
5. $5/4$,	6. $7/4$,	7. -2,	8. 0,
9. 0,	10. 6,	11. $1/9$,	12. $-7/6$,
13. $23/5$,	14. 6,	15. -5,	16. $37/19$,
17. -30,	18. $3/4$,	19. $-5/3$,	20. -5,
21. 7,	22. $\sqrt{8} - 4$,	23. $\sqrt{23} + 4$,	24. -49,
25. $12/19$,	26. 42,	27. 1,	28. $8/13$,
29. $-7/3$,	30. $13/3$,	31. 15,	32. $7/6$,
33. $-2/5$,	34. $x = 9/200$.	35. $s = 13$.	

Section 7. Transposition of formulae

1. $t = \frac{c^2}{p^2}$, 2. $N = \sqrt{\frac{L\ell}{\mu A}}$,
3. a) $e = \frac{h-c-d}{2}$, b) $h = \frac{S-2\pi r^2}{2\pi r}$, c) $c = \frac{d(1+Q^2)}{Q^2-1}$ d) $x = \frac{21-5y}{2}$.
4. $n = \frac{mJ}{E-LJ}$

Section 8. Solving quadratic equations by factorisation

1. 1,2,	2. $-1, 2$,	3. $-2, 1$,	4. $-1, -2$,
5. $-7, -1$,	6. $4, 3$,	7. $-4, 5$,	8. $1, -1$,
9. 1 twice,	10. -1 twice	11. $-11, 0$,	12. $0, -1$,
13. 0,3,	14. $0, -9$,	15. $2, \frac{1}{2}$,	16. $\frac{1}{2}$, $-\frac{1}{3}$,
17. $\frac{1}{5}$,1.	18. $1, 3$.		

Section 9. Solving quadratic equations by using a standard formula and by completing the square

Note that answers were requested in surd form. Decimal approximations are not acceptable.

1. $-4 \pm \sqrt{15}$.	2. $-\frac{7}{2} \pm \frac{\sqrt{57}}{2}$.	3. $-3 \pm \sqrt{11}$.	4. $-\frac{3}{8} \pm \frac{\sqrt{41}}{8}$.
5. $-\frac{3}{4} \pm \frac{\sqrt{17}}{4}$.	6. $-\frac{1}{2} \pm \frac{\sqrt{5}}{2}$.	7. $\frac{3}{2} \pm \frac{\sqrt{13}}{2}$.	8. $-\frac{3}{4} \pm \frac{\sqrt{17}}{4}$.
9. $\frac{1}{2}, -3$.	10. $-3/2, 1$.	11. $-\frac{8}{9} \pm \frac{\sqrt{55}}{9}$.	12. $-8 \pm \sqrt{55}$.

Section 10. Solving some polynomial equations

1. $(x+7)(x+1)(x-9)$, 2. $x=-1,-3,-7$,
3. $x=2,-1$. 4. $x=-1,1$.
5. $(x+1)^2(x-4)(x-5)$. 6. $(x+3)(x-3)(x+1)(x+2)$

Section 11. Partial fractions

1. a) $\frac{2}{x+1}+\frac{3}{x-2}$, c) $\frac{3}{x+4}+\frac{4}{x+3}$
2. $\frac{4}{x-1}+\frac{3}{2x+1}$, 3. a) $\frac{3}{x+1}-\frac{3}{x+2}$, b) $\frac{5}{x+3}-\frac{5}{x+4}$, c) $\frac{6}{7(2x+1)}-\frac{3}{7(x-3)}$.
4. a) $-\frac{1}{x-1}+\frac{2}{(x-1)^2}$, b) $-\frac{7}{x-1}+\frac{8}{(x-1)^2}$, c) $\frac{3}{x+4}+\frac{2}{(x+4)^2}$.
d) $\frac{5}{x+4}-\frac{2}{(x+4)^2}$, e) $\frac{1}{x+1}+\frac{1}{x-1}+\frac{1}{(x-1)^2}$, f) $\frac{3}{2x+3}+\frac{1}{x+2}+\frac{2}{(x+2)^2}$,
g) $-\frac{1}{3x-2}+\frac{1}{(3x-2)^2}+\frac{1}{x+7}$ h) $\frac{1}{s+1}+\frac{1}{(s+1)^2}$ i) $\frac{2}{s}+\frac{3}{s^2}$.

5. a) $\frac{3}{7(x-2)}-\frac{3(x+3)}{7(x^2+x+1)}$
b) $\frac{3x+1}{6x^2+x+2}+\frac{4}{x-3}$
c) $\frac{1}{2x+3}+\frac{1}{(2x+3)^2}$
d) $\frac{3x+1}{x^2+5x+1}+\frac{3}{x-1}$.
6. a) $1+\frac{1}{x+2}$
b) $3+\frac{2}{x-3}$,
c) $1+x+\frac{1}{x+1}$
d) $2x+3+\frac{1}{x+2}$,
e) $\frac{1}{(x+2)^2}+\frac{1}{x+2}+\frac{1}{x-3}+3x^2+x+2$
f) $2x^2+x+7+\frac{1}{2x+1}+\frac{1}{x^2+x+1}$
7. a) $-\frac{4}{3x}+\frac{1}{x-1}+\frac{1}{3(x-3)}$. b) $\frac{1}{(x+3)^2}$. c) $\frac{5}{21(2x+1)}-\frac{1}{3(x-1)}+\frac{5}{7(x-3)}$
d) $2-\frac{5}{2s+1}$ e) $\frac{7}{2(s-2)}-\frac{1}{2s}$.
8. $\frac{K}{s}+\frac{K(\alpha-\tau)}{1+\tau s}$
9. a) $\frac{1}{s^5}+\frac{1}{s^4}-\frac{1}{s^3}+\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}$. b) $\frac{1}{s+1}+2s^2+4s+2$.
10. $-\frac{5}{3(x-2)}-\frac{1}{12(x+1)}+\frac{7}{4(x-3)}$

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Further details from:
mathcentre
c/o Mathematics Education Centre
Loughborough University
LE11 3TU

email: enquiries@mathcentre.ac.uk
tel: 01509 227465
www: http://www.mathcentre.ac.uk