

The vector product

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One of the ways in which two vectors can be combined is known as the **vector product**. When we calculate the vector product of two vectors the result, as the name suggests, is a vector.

In this unit you will learn how to calculate the vector product and meet some geometrical applications.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- define the vector product of two vectors
- calculate the vector product when the two vectors are given in cartesian form
- use the vector product in some geometrical applications

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1. Introduction

One of the ways in which two vectors can be combined is known as the **vector product**. When we calculate the vector product of two vectors the result, as the name suggests, is a vector.

In this unit you will learn how to calculate the vector product and meet some geometrical applications.

2. Definition of the vector product

Study the two vectors **a** and **b** drawn in Figure 1. Note that we have drawn the two vectors so that their tails are at the same point. The angle between the two vectors has been labelled θ .

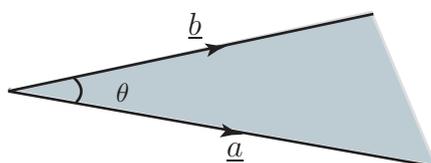


Figure 1. Two vectors **a** and **b** drawn so that the angle between them is θ .

As we stated before, when we find a **vector product** the result is a vector. We define the modulus, or magnitude, of this vector as

$$|\mathbf{a}| |\mathbf{b}| \sin \theta$$

so at this stage, a very similar definition to the scalar product, except now the sine of θ appears in the formula. However, this quantity is not a vector. To obtain a vector we need to specify a direction. By definition the direction of the vector product is such that it is at right angles to both **a** and **b**. This means it is at right angles to the plane in which **a** and **b** lie. Figure 2 shows that we have two choices for such a direction.

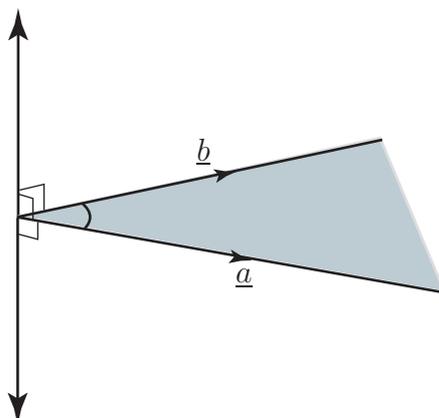


Figure 2. There are two directions which are perpendicular to both **a** and **b**.

The convention is that we choose the direction specified by the right hand screw rule. This means that we imagine a screwdriver in the right hand. The direction of the vector product is

the direction in which a screw would advance as the screwdriver handle is turned in the sense from \mathbf{a} to \mathbf{b} . This is shown in Figure 3.

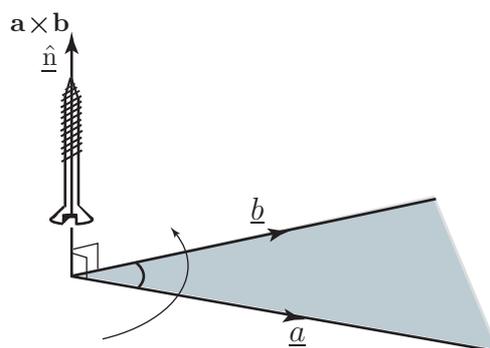


Figure 3. The direction of the vector product is determined by the right hand screw rule. We let a unit vector in this direction be labelled $\hat{\mathbf{n}}$. We then define the vector product of \mathbf{a} and \mathbf{b} as follows:



Key Point

The **vector product** of \mathbf{a} and \mathbf{b} is defined to be

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin \theta \hat{\mathbf{n}}$$

where

$|\mathbf{a}|$ is the modulus, or magnitude of \mathbf{a} ,

$|\mathbf{b}|$ is the modulus of \mathbf{b} ,

θ is the angle between \mathbf{a} and \mathbf{b} , and $\hat{\mathbf{n}}$ is a unit vector, perpendicular to both \mathbf{a} and \mathbf{b} in a sense defined by the right hand screw rule.

Some people find it helpful to obtain the direction of the vector product using the right hand thumb rule. This is achieved by curling the fingers of the right hand in the direction in which \mathbf{a} would be rotated to meet \mathbf{b} . The thumb then points in the direction of $\mathbf{a} \times \mathbf{b}$.

Yet another view is to align the first finger of the right hand with \mathbf{a} , and the middle finger with \mathbf{b} . If these two fingers and the thumb are then positioned at right-angles, the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$. Try this for yourself.

Note that the symbol for the vector product is the times sign, or cross \times , and so we sometimes refer to the vector product as the cross product. Either name will do. Some textbooks and some teachers and lecturers use the alternative 'wedge' symbol \wedge .

3. Some properties of the vector product

Suppose, for the two vectors \mathbf{a} and \mathbf{b} we calculate the product in a different order. That is, suppose we want to find $\mathbf{b} \times \mathbf{a}$. Using the definition of $\mathbf{b} \times \mathbf{a}$ and using the right-hand screw rule to obtain the required direction we find

$$\mathbf{b} \times \mathbf{a} = |\mathbf{b}| |\mathbf{a}| \sin \theta (-\hat{\mathbf{n}})$$

We see that the direction of $\mathbf{b} \times \mathbf{a}$ is opposite to that of $\mathbf{a} \times \mathbf{b}$ as shown in Figure 4. So

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

So the vector product is **not commutative**. In practice, this means that the order in which we do the calculation does matter. $\mathbf{b} \times \mathbf{a}$ is in the opposite direction to $\mathbf{a} \times \mathbf{b}$.

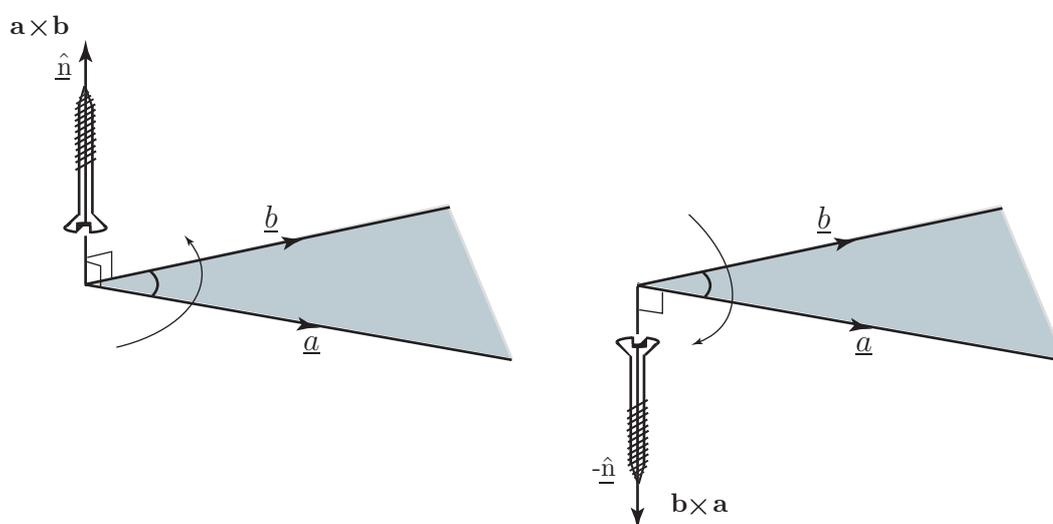


Figure 4. The direction of $\mathbf{b} \times \mathbf{a}$ is opposite to that of $\mathbf{a} \times \mathbf{b}$.



Key Point

The vector product is **not commutative**.

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

Another property of the vector product is that it is **distributive over addition**. This means that

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

Although we shall not prove this result here we shall use it later on when we develop an alternative formula for finding the vector product.



Key Point

The vector product is **distributive over addition**. This means

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

Equivalently,

$$(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$$

The vector product of two parallel vectors

Example

Suppose the two vectors \mathbf{a} and \mathbf{b} are parallel. Strictly speaking the definition of the vector product does not apply, because two parallel vectors do not define a plane, and so it does not make sense to talk about a unit vector \hat{n} perpendicular to the plane. But if we nevertheless write down the formula, we can see what the answer 'ought' to be:

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= |\mathbf{a}| |\mathbf{b}| \sin \theta \hat{n} \\ &= |\mathbf{a}| |\mathbf{b}| \sin 0^\circ \hat{n} \\ &= \mathbf{0}\end{aligned}$$

because $\sin 0^\circ = 0$. So, when two vectors are parallel we *define* their vector product to be the zero vector, $\mathbf{0}$.



Key Point

For two parallel vectors

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

4. The vector product of two vectors given in cartesian form

We now consider how to find the vector product of two vectors when these vectors are given in cartesian form, for example as

$$\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k} \quad \text{and} \quad \mathbf{b} = -5\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$$

where \mathbf{i} , \mathbf{j} and \mathbf{k} are unit vectors in the directions of the x , y and z axes respectively.

First of all we need to develop a few results in the following examples.

Example

Suppose we want to find $\mathbf{i} \times \mathbf{j}$. The vectors \mathbf{i} and \mathbf{j} are shown in Figure 5. Note that because these vectors lie along the x and y axes they must be perpendicular.

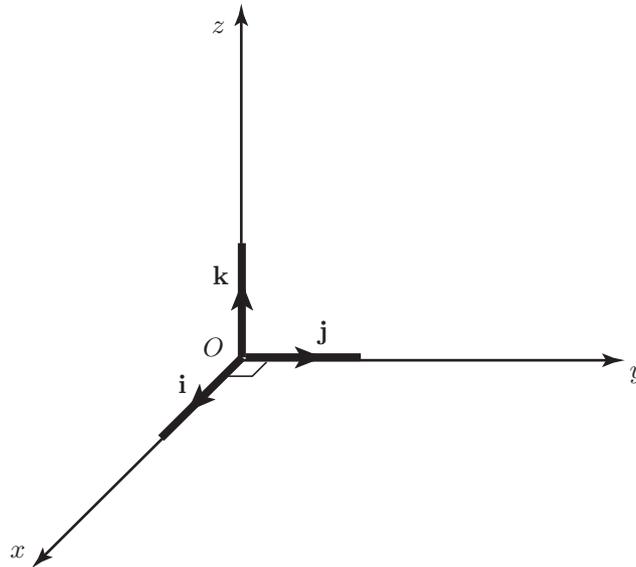


Figure 5. The unit vectors \mathbf{i} , \mathbf{j} and \mathbf{k} . Note that \mathbf{k} is a unit vector perpendicular to \mathbf{i} and \mathbf{j} . The angle between \mathbf{i} and \mathbf{j} is 90° , and $\sin 90^\circ = 1$. Further, if we apply the right hand screw rule, a vector perpendicular to both \mathbf{i} and \mathbf{j} is \mathbf{k} . Therefore

$$\begin{aligned}\mathbf{i} \times \mathbf{j} &= |\mathbf{i}| |\mathbf{j}| \sin 90^\circ \mathbf{k} \\ &= (1)(1)(1) \mathbf{k} \\ &= \mathbf{k}\end{aligned}$$

Example

Suppose we want to find $\mathbf{j} \times \mathbf{i}$. Again, refer to Figure 5. If we apply the right hand screw rule, a vector perpendicular to both \mathbf{j} and \mathbf{i} , in the sense defined by the right hand screw rule, is $-\mathbf{k}$. Therefore

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

Example

Suppose we want to find $\mathbf{i} \times \mathbf{i}$. Because these two vectors are parallel the angle between them is 0° . We can use the Key Point developed on page 5 to show that $\mathbf{i} \times \mathbf{i} = \mathbf{0}$.

In a similar manner we can derive all the results given in the following Key Point:



Key Point

$$\begin{array}{lll}\mathbf{i} \times \mathbf{i} = \mathbf{0} & \mathbf{j} \times \mathbf{j} = \mathbf{0} & \mathbf{k} \times \mathbf{k} = \mathbf{0} \\ \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j}\end{array}$$

We can use these results to develop a formula for finding the vector product of two vectors given in cartesian form:

Suppose $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\begin{aligned}
 \mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
 &= a_1\mathbf{i} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
 &\quad + a_2\mathbf{j} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
 &\quad + a_3\mathbf{k} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
 &= a_1\mathbf{i} \times b_1\mathbf{i} + a_1\mathbf{i} \times b_2\mathbf{j} + a_1\mathbf{i} \times b_3\mathbf{k} \\
 &\quad + a_2\mathbf{j} \times b_1\mathbf{i} + a_2\mathbf{j} \times b_2\mathbf{j} + a_2\mathbf{j} \times b_3\mathbf{k} \\
 &\quad + a_3\mathbf{k} \times b_1\mathbf{i} + a_3\mathbf{k} \times b_2\mathbf{j} + a_3\mathbf{k} \times b_3\mathbf{k} \\
 &= a_1b_1\mathbf{i} \times \mathbf{i} + a_1b_2\mathbf{i} \times \mathbf{j} + a_1b_3\mathbf{i} \times \mathbf{k} \\
 &\quad + a_2b_1\mathbf{j} \times \mathbf{i} + a_2b_2\mathbf{j} \times \mathbf{j} + a_2b_3\mathbf{j} \times \mathbf{k} \\
 &\quad + a_3b_1\mathbf{k} \times \mathbf{i} + a_3b_2\mathbf{k} \times \mathbf{j} + a_3b_3\mathbf{k} \times \mathbf{k}
 \end{aligned}$$

Now, from the previous Key Point three of these terms are zero. Those that are not zero simplify to give

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

This is the formula which we can use to calculate a vector product when we are given the cartesian components of the two vectors.



Key Point

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Example

Suppose we wish to find the vector product of the two vectors $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$.

We use the previous result with $a_1 = 4$, $a_2 = 3$, $a_3 = 7$ and $b_1 = 2$, $b_2 = 5$, $b_3 = 4$. Substitution into the formula gives

$$\mathbf{a} \times \mathbf{b} = ((3)(4) - (7)(5))\mathbf{i} + ((7)(2) - (4)(4))\mathbf{j} + ((4)(5) - (3)(2))\mathbf{k}$$

which simplifies to

$$\mathbf{a} \times \mathbf{b} = -23\mathbf{i} - 2\mathbf{j} + 14\mathbf{k}$$

For those familiar with evaluation of **determinants** there is a convenient way of remembering and representing this formula which is given in the following Key Point and which is explained in the accompanying video and in the Example below.



Key Point

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \\ &= (a_2 \times b_3 - a_3 \times b_2)\mathbf{i} - (a_1 \times b_3 - a_3 \times b_1)\mathbf{j} + (a_1 \times b_2 - a_2 \times b_1)\mathbf{k} \end{aligned}$$

Example

Suppose we wish to find the vector product of the two vectors $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$.

We write down a determinant, which is an array of numbers: in the first row we write the three unit vectors \mathbf{i} , \mathbf{j} and \mathbf{k} . In the second and third rows we write the three components of \mathbf{a} and \mathbf{b} respectively:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 7 \\ 2 & 5 & 4 \end{vmatrix}$$

We then consider the first element in the first row, \mathbf{i} . Imagine covering up the elements in its row and column, to give the array $\begin{vmatrix} 3 & 7 \\ 5 & 4 \end{vmatrix}$. This is a so-called 2×2 determinant and is evaluated by finding the product of the elements on the leading diagonal (top left to bottom right) and subtracting the product of the elements on the other diagonal ($3 \times 4 - 7 \times 5 = -23$). The resulting number gives the \mathbf{i} component of the final answer.

We then consider the second element in the first row, \mathbf{j} . Imagine covering up the elements in its row and column, to give the array $\begin{vmatrix} 4 & 7 \\ 2 & 4 \end{vmatrix}$. This 2×2 determinant is evaluated, as before, by finding the product of the elements on the leading diagonal (top left to bottom right) and subtracting the product of the elements on the other diagonal, ($4 \times 4 - 7 \times 2 = 2$). The result is then multiplied by -1 and this gives the \mathbf{j} component of the final answer, that is -2 .

Finally, we consider the third element in the first row, \mathbf{k} . Imagine covering up the elements in its row and column, to give the array $\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}$. This determinant is evaluated, as before, by finding

the product of the elements on the leading diagonal (top left to bottom right) and subtracting the product of the elements on the other diagonal ($4 \times 5 - 3 \times 2 = 14$). The resulting number gives the \mathbf{k} component of the final answer.

We write all this as follows:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 7 \\ 2 & 5 & 4 \end{vmatrix} \\ &= \begin{vmatrix} 3 & 7 \\ 5 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4 & 7 \\ 2 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix} \mathbf{k} \\ &= (3 \times 4 - 7 \times 5)\mathbf{i} - (4 \times 4 - 7 \times 2)\mathbf{j} + (4 \times 5 - 3 \times 2)\mathbf{k} \\ &= -23\mathbf{i} - 2\mathbf{j} + 14\mathbf{k} \end{aligned}$$

Exercises 1

1. Use the formula $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$ to find the vector product $\mathbf{a} \times \mathbf{b}$ in each of the following cases.

(a) $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = -2\mathbf{i} + 9\mathbf{j}$.

(b) $\mathbf{a} = 4\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = 5\mathbf{i} - 7\mathbf{j}$.

Comment upon your solutions.

2. Use the formula in Q1 to find the vector product $\mathbf{a} \times \mathbf{b}$ in each of the following cases.

(a) $\mathbf{a} = 5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 8\mathbf{j} + 9\mathbf{k}$.

(b) $\mathbf{a} = \mathbf{i} + \mathbf{j} - 12\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

3. Use determinants to find the vector product $\mathbf{p} \times \mathbf{q}$ in each of the following cases.

(a) $\mathbf{p} = \mathbf{i} + 4\mathbf{j} + 9\mathbf{k}$, $\mathbf{q} = 2\mathbf{i} - \mathbf{k}$.

(b) $\mathbf{p} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{q} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$.

4. For the vectors $\mathbf{p} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{q} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$ show that, in this special case, $\mathbf{p} \times \mathbf{q} = \mathbf{q} \times \mathbf{p}$.

5. For the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{c} = 7\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, show that

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$$

5. Some applications of the vector product

In this section we will look at some ways in which the vector product can be used.

Using the vector product to find a vector perpendicular to two given vectors.

One of the common applications of the vector product is to finding a vector which is perpendicular to two given vectors. The two vectors should be non-zero and must not be parallel.

Example

Suppose we wish to find a vector which is perpendicular to both of the vectors $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} - 3\mathbf{k}$.

We know from the definition of the vector product that the vector $\mathbf{a} \times \mathbf{b}$ will be perpendicular to both \mathbf{a} and \mathbf{b} . So first of all we calculate $\mathbf{a} \times \mathbf{b}$.

$$\begin{aligned}
 \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 5 & 0 & -3 \end{vmatrix} \\
 &= (3 \times -3 - (-2) \times 0)\mathbf{i} - (1 \times -3 - (-2) \times 5)\mathbf{j} + (1 \times 0 - 3 \times 5)\mathbf{k} \\
 &= -9\mathbf{i} - 7\mathbf{j} - 15\mathbf{k}
 \end{aligned}$$

This vector is perpendicular to \mathbf{a} and \mathbf{b} .

On occasions you may be asked to find a unit vector which is perpendicular to two given vectors. To convert a vector into a unit vector in the same direction we must divide it by its modulus. The modulus of $-9\mathbf{i} - 7\mathbf{j} - 15\mathbf{k}$ is

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{(-9)^2 + (-7)^2 + (-15)^2} = \sqrt{355}$$

So, finally, the required unit vector is $\frac{1}{\sqrt{355}}(-9\mathbf{i} - 7\mathbf{j} - 15\mathbf{k})$.

Using the vector product to find the area of a parallelogram.

Consider the parallelogram shown in Figure 6 which has sides given by vectors \mathbf{b} and \mathbf{c} .

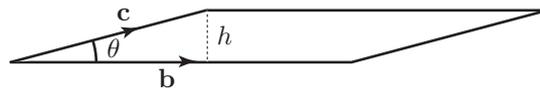


Figure 6. A parallelogram with two sides given by \mathbf{b} and \mathbf{c} .

The area of the parallelogram is the length of the base multiplied by the perpendicular height, h .

Now $\sin \theta = \frac{h}{|\mathbf{c}|}$ and so $h = |\mathbf{c}| \sin \theta$. Therefore

$$\text{area} = |\mathbf{b}| |\mathbf{c}| \sin \theta$$

which is simply the modulus of the vector product of \mathbf{b} and \mathbf{c} . We deduce that the area of the parallelogram is given by

$$\text{area} = |\mathbf{b} \times \mathbf{c}|$$

Using the vector product to find the volume of a parallelepiped.

Consider Figure 7 which illustrates a parallelepiped. This is a six sided solid, the sides of which are parallelograms. Opposite parallelograms are identical. The volume, V , of a parallelepiped with edges \mathbf{a} , \mathbf{b} and \mathbf{c} is given by

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

This formula can be obtained by understanding that the volume is the product of the area of the base and the perpendicular height. Because the base is a parallelogram its area is $|\mathbf{b} \times \mathbf{c}|$. The perpendicular height is the component of \mathbf{a} in the direction perpendicular to the plane containing \mathbf{b} and \mathbf{c} , and this is $h = \mathbf{a} \cdot \widehat{\mathbf{b} \times \mathbf{c}}$. So the volume is given by

$$\begin{aligned}
 V &= (\text{height})(\text{area of base}) \\
 &= \mathbf{a} \cdot \widehat{\mathbf{b} \times \mathbf{c}} |\mathbf{b} \times \mathbf{c}| \\
 &= \mathbf{a} \cdot \frac{\mathbf{b} \times \mathbf{c}}{|\mathbf{b} \times \mathbf{c}|} |\mathbf{b} \times \mathbf{c}| \\
 &= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})
 \end{aligned}$$

This could turn out to be negative, so in fact, for the volume we take its modulus: $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

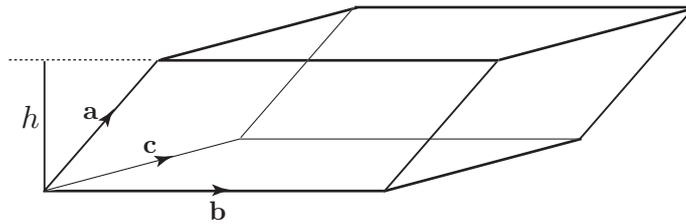


Figure 7. A parallelepiped with edges given by \mathbf{a} , \mathbf{b} and \mathbf{c} .

Example

Suppose we wish to find the volume of the parallelepiped with edges $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$.

We first evaluate the vector product $\mathbf{b} \times \mathbf{c}$.

$$\begin{aligned} \mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix} \\ &= (1 \times 4 - 1 \times 2)\mathbf{i} - (2 \times 4 - 1 \times 1)\mathbf{j} + (2 \times 2 - 1 \times 1)\mathbf{k} \\ &= 2\mathbf{i} - 7\mathbf{j} + 3\mathbf{k} \end{aligned}$$

Then we need to find the scalar product of \mathbf{a} with $\mathbf{b} \times \mathbf{c}$.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (3\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} - 7\mathbf{j} + 3\mathbf{k}) = 6 - 14 + 3 = -5$$

Finally, we want the modulus, or absolute value, of this result. We conclude the parallelepiped has volume 5 (units cubed).

Exercises 2.

1. Find a unit vector which is perpendicular to both $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.
2. Find the area of the parallelogram with edges represented by the vectors $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $7\mathbf{i} + \mathbf{j} + \mathbf{k}$.
3. Find the volume of the parallelepiped with edges represented by the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.
4. Calculate the **triple scalar product** $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ when $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

Answers to Exercises

Exercises 1.

1. (a) $24\mathbf{k}$, (b) $-18\mathbf{k}$. Both answers are vectors in the z direction. The given vectors, \mathbf{a} and \mathbf{b} , lie in the xy plane.
2. (a) $59\mathbf{i} - 37\mathbf{j} - 46\mathbf{k}$, (b) $13\mathbf{i} - 25\mathbf{j} - \mathbf{k}$.
3. (a) $-4\mathbf{i} + 19\mathbf{j} - 8\mathbf{k}$, (b) $-\mathbf{i} + 10\mathbf{j} - 7\mathbf{k}$.
4. Both cross products equal zero, and so, in this special case $\mathbf{p} \times \mathbf{q} = \mathbf{q} \times \mathbf{p}$. The two given vectors are anti-parallel.
5. Both equal $-11\mathbf{i} + 25\mathbf{j} - 13\mathbf{k}$.

Exercises 2.

1. $\frac{1}{\sqrt{171}}(11\mathbf{i} - 7\mathbf{j} - \mathbf{k})$.
2. $\sqrt{458}$ square units.
3. 8 units cubed.
4. 7.