Numerical Quadrature

mccp-Simpson-001

For simple functions we can use definite integration to find the area under the curve on the graph, however, when looking at more complex functions we can often only estimate the integrated value, usually by splitting the area under the graph into smaller sections and calculating their areas seperately. This is done by means of one of many viable formulae and is known as *Numerical Quadrature*. Numerical Quadrature refers to any method used to numerically estimate the value of a definite integral. On this sheet we will be looking at two of the most commonly used methods of doing this.

Trapezium rule

To find the area underneath a curve between the points a and b using the *Trapezium rule* we first split it into smaller intervals, each with width h. Where $h = \frac{b-a}{n}$ and n is the number of intervals. This has the effect of splitting it into a number of narrow trapeziums. The equation for the *Trapezium rule* is:

$$\int_a^b f(x)\,\mathrm{d}x = \frac{1}{2}h[(f(x_0) + f(x_n)) + 2(f(x_1) + f(x_2) + \ldots + f(x_{n-1})]$$

Where $f_0 =$ your starting point (a), $f_1 =$ the first interval after that point, $f_2 =$ the second, and so on until $f_n =$ your finishing point (b).

Example

Use the Trapezium rule with 4 intervals to calculate an estimate for the definite integral of

$$\int_0^2 \sqrt{3x+1}\,\mathrm{d}x$$

Solution

In this case $h = 0.5$, therefore $x_0 = 0$, $f_1 = 0.5$, $f_2 = 1$, $f_3 = 1.5$ and $f_4 = 2$. We can substitute these values into the equation:

$$\int_0^2 \sqrt{3x+1}\,\mathrm{d}x = \frac{1}{2}\cdot 0.5[(\sqrt{1}+\sqrt{7}) + 2(\sqrt{2.5}+\sqrt{4}+\sqrt{5.5})] = 3.8746111827641$$

(or 3.87 to 3 significant figures)

If the same integral was estimated using more intervals the answer would be more accurate, for example using 8 intervals, so with an interval size of 0.25 gives the answer 3.8885807826407 or 3.89 to 3 significant figures. Using 16 intervals gives 3.8921793289127 and using 32 intervals gives 3.893087411421. So as you can see the answers are tending towards approximately 3.893 as more intervals are being used.

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Simpson's rule

Simpson's rule works in theory in much the same way as the Trapezium rule, except these narrow trapeziums are topped with parabolas as opposed to straight line segments. Each parabola requires three points to specify it, so spans two intervals. Given that this method can only be used once and must utilise complete parabolas the number of intervals must always be even. The distance between these intervals is also given as $\frac{b-a}{n}$ and the equation to produce Simpson's rule is:

$$\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{3n}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\ldots+4f(x_{n-1})+f(x_n)]$$

Example

Use Simpson's rule with 6 intervals to estimate the definite intergal of

$$\int_1^7 \sqrt{x^3-x}\,\mathrm{d}x$$

Solution

The first thing to do is work out the interval size using $\frac{b-a}{n}$ so in this case $\frac{7-1}{6}=1$.

Now we just need to substitute these intervals and $\int_1^7 \sqrt{x^3-x}\,\mathrm{d}x$ into our equation, giving us:

$$\int_1^7 \sqrt{x^3-x}\,\mathrm{d}x = \frac{7-1}{6\cdot 3}[\sqrt{0}+4\sqrt{6}+2\sqrt{24}+4\sqrt{60}+2\sqrt{120}+4\sqrt{210}+\sqrt{336}] = 49.594832258904$$

(or 49.6 to 3 significant figures)

Error function

e^{-x^2} is a good example to use for numerical integration because there is no simple function which has it as a derivative. It is also part of a very important function; $\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t$ is known as the *error function*, which has uses mostly in probability and statistcs, specifically normal distibution and estimations of standard deviation, but also stretches as far as partial differential equations.

Example

Use Simspon's rule with 4 intervals to estimate the definite integral of

$$\frac{2}{\sqrt{\pi}}\int_0^1 e^{-t^2}\,\mathrm{d}t$$

Solution

In this case the interval size is 0.25. This allows us to produce the formula:

$$\frac{2}{\sqrt{\pi}}\int_0^1 e^{-t^2}\,\mathrm{d}t = \frac{2}{\sqrt{\pi}}\cdot\frac{1-0}{3\cdot 4}[e^{-0^2}+4(e^{-0.25^2})+2(e^{-0.5^2})+4(e^{-0.75^2})+e^{-1^2}] = 0.84273605138$$

(or 0.843 to 3 significant figures)

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