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The inverse of a 2×2 matrix

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Once you know how to multiply matrices it is natural to ask whether they can be divided. The answer is no. However, by defining another matrix called the **inverse matrix** it is possible to work with an operation which plays a similar role to division. In this leaflet we explain what is meant by an inverse matrix and how the inverse of a 2×2 matrix is calculated.

Preliminary example

Suppose we calculate the product of the two matrices $\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$:

$$\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If we re-order the matrices and recalculate we will obtain the same result. You should verify this:

$$\begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that the result of multiplying the two matrices together is the **identity** matrix. Pairs of square matrices which have this property are called **inverse** matrices. The first is the inverse of the second, and vice-versa.

The inverse of a 2×2 matrix

The **inverse** of a 2×2 matrix A, is another 2×2 matrix denoted by A^{-1} with the property that

$$AA^{-1} = A^{-1}A = I$$

where I is the 2×2 identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. That is, multiplying a matrix by its inverse produces an identity matrix. Note that in this context A^{-1} does not mean $\frac{1}{A}$.

Not all 2×2 matrices have an inverse matrix. If the determinant of the matrix is zero, then it will not have an inverse; the matrix is then said to be **singular**. Only non-singular matrices have inverses.

A simple formula for the inverse

In the case of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a simple formula exists to find its inverse:

$$\text{if} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{then} \quad A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Note that the quantity $ad - bc$ is the determinant of A. Furthermore, $\frac{1}{ad-bc}$ is not defined when $ad - bc = 0$ since it is never possible to divide by zero. It is for this reason that the inverse of A does not exist if the determinant of A is zero.

Example

Find the inverse of the matrix $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$.

Solution

Using the formula

$$\begin{aligned} A^{-1} &= \frac{1}{(3)(2)-(1)(4)}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} \end{aligned}$$

This could be written as

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{pmatrix}$$

You should check that this answer is correct by performing the matrix multiplication AA^{-1}. The result should be the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Example

Find the inverse of the matrix $A = \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}$.

Solution

Using the formula

$$\begin{aligned} A^{-1} &= \frac{1}{(2)(1)-(4)(-3)}\begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix} \\ &= \frac{1}{14}\begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix} \end{aligned}$$

This can be written

$$A^{-1} = \begin{pmatrix} 1/14 & -4/14 \\ 3/14 & 2/14 \end{pmatrix} = \begin{pmatrix} 1/14 & -2/7 \\ 3/14 & 1/7 \end{pmatrix}$$

although it is quite permissible to leave the factor $\frac{1}{14}$ at the front of the matrix.

Example

Find, if possible, the inverse of the matrix $A = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$.

Solution

In this case the determinant of the matrix is zero:

$$\begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix} = 3 \times 4 - 2 \times 6 = 0$$

Because the determinant is zero the matrix is singular and no inverse exists.

We explain how to find the inverse of a 3×3 matrix in a later leaflet in this series.

Note that a video tutorial covering the content of this leaflet is available from **sigma**.

